J. Fluid Mech. (1978), *vol.* 86, *part* 2, *pp.* 395–400
Printed in Great Britain

On the applicability of Howard's formula

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(Received 19 January 1977)

The applicability of Howard's formula is considered in cases where there exist critical layers at the boundaries in addition to one inside the flow field. It is shown that in a few particular cases Howard's formula can be applied provided that the finite part of an otherwise diverging integral is taken. In general, however, this revised formula is not valid and an expression analogous to the one given by Banks, Drazin & Zaturska for a particular example is found.

1. Introduction

In a paper by Huppert (1973) the validity of the formula which Howard developed in connexion with perturbing neutral solutions of the Taylor–Goldstein equation is considered. Among the examples which he considers there is one with a non-monotonic velocity profile in which there are critical layers at the boundaries in addition to the one inside the flow field. It is easily seen that one of the integrals which enters Howard's formula does not exist in the ordinary sense in this case. However, it seems to us that Huppert adopts (without saying so) the rule that the finite part of that integral is to be taken, and applies Howard's formula with this rule in mind. The rule of taking the finite part of the integrals is correct in cases where the critical layers are situated within the flow field (see Howard 1963; or Engevik 1973). In a recent paper Banks, Drazin & Zaturska (1976) re-examined this example of Huppert. They solve the eigenvalue problem numerically and find that on part of the stability boundary Huppert's result does not give the correct value of c_i (the imaginary part of c, the wave velocity). On this part of the stability boundary Banks *et al.* obtain an expression for c_i, based on Howard's formula, which differs from the one given by Huppert. They find that the dominant contribution to one of the integrals comes from the regions near the boundaries, and this contribution is not taken into account by Huppert. The values of c_i obtained from their theoretical expression are found to be in good agreement with their numerical results. However, since their expression for c_i is based on Howard's formula, they conclude correctly that 'the logical justification for the result is poor' (p. 164).

The purpose of this note is twofold. First, we show that the rule of taking the finite part of an (otherwise diverging) integral may be correct in some cases, and that in Huppert's example this happens to be true on part of the stability boundary. Second, we prove the expression for c_i obtained by Banks *et al.* to be correct.

Our investigation is based on a method given in a paper by Engevik (1973).

2. The general theory

Let the stream function corresponding to an infinitesimal disturbance in a parallel, two-dimensional, inviscid, heterogeneous shear flow be denoted by $\phi(z)\, e^{ik(x-ct)}$. If the Boussinesq approximation is made, the amplitude function $\phi(z)$ satisfies the Taylor–Goldstein equation

$$\phi'' + \left[\frac{JN^2}{(U-c)^2} - \frac{U''}{U-c} - k^2\right]\phi = 0. \tag{2.1}$$

Here $U(z)$ is the basic flow velocity, $N(z)$ the buoyancy frequency, J a (representative) Richardson number, k the wavenumber and c the wave velocity (which may be complex). If the fluid is confined between two rigid horizontal planes at $z = z_1,\, z_2$ the boundary conditions are

$$\phi = 0 \quad \text{at} \quad z = z_1, z_2. \tag{2.2}$$

In this note we assume that there is just one critical layer within the flow field, situated at $z = z_c$, and that $U'(z_c) > 0$. However there may be further critical layers at the boundaries.

Let a neutral solution be denoted by ϕ_s and the corresponding wavenumber, Richardson number and wave velocity by k_s, J_s and c_s respectively. The neutral solutions may have contiguous unstable solutions, but there also exist situations where this is not true as shown at the end of this section; examples of this are given by Huppert (1973). Now let ϕ denote an unstable solution contiguous to ϕ_s and proceed as in Engevik (1973) to obtain

$$(I_1 - I_2)(c-c_s) + I_3(J-J_s) - I_4(k^2-k_s^2) = 0, \tag{2.3}$$

where
$$\left.\begin{aligned} I_1 &= \int_L \frac{J_s N^2\{2U-(c+c_s)\}\phi\phi_s}{(U-c_s)^2(U-c)^2}dz, \quad I_2 = \int_L \frac{U''\phi\phi_s}{(U-c_s)(U-c)}dz,\\ I_3 &= \int_L \frac{N^2\phi\phi_s}{(U-c)^2}dz, \quad I_4 = \int_L \phi\phi_s dz. \end{aligned}\right\} \tag{2.4}$$

The integration is from z_1 to z_2 along a contour L which circumvents the critical point at z_c in the correct way (Engevik 1973). Analyticity of U and N^2 for $z_1 \leqslant z \leqslant z_2$ is assumed, and $\arg(U-c_s)$ is defined to be zero for $U-c_s > 0$ and $-\pi$ for $U-c_s < 0$.

If there are no critical layers at the boundaries, we can use (2.3) to derive Howard's formula for $(\partial c/\partial k^2)_{J_s}$ and a formula for $(\partial c/\partial J)_{k_s}$. If the denominator in these formulae (obtained by putting $c = c_s$ and $\phi = \phi_s$ in $I_1 - I_2$) becomes zero, we have to expand (2.3) to take into account the term in $(c-c_s)^2$. Then formulae for

$$(\partial(c-c_s)^2/\partial k^2)_{J_s} \quad \text{and} \quad (\partial(c-c_s)^2/\partial J)_{k_s}$$

can be found (Engevik 1973, 1975). In a particular case with a given velocity and density profile Banks & Drazin (1973) have found the expression for $(\partial(c-c_s)^2/\partial k^2)_{J_s}$ essentially along these lines.

In the following we assume that there are critical layers at the boundaries in addition to the one within the flow field and that $U' \neq 0$ at $z = z_1, z_2$. If we assume also that N is not equal to zero at $z = z_1, z_2$ then Howard's formula is not valid, because at least one of the integrals in this formula does not exist. However, (2.3) is still valid and from this equation we can find $c - c_s$ as a function of $k^2 - k_s^2$ when $k \to k_s$ with $J = J_s$ fixed or as a function of $J - J_s$ when $J \to J_s$ with $k = k_s$ fixed. We give the general procedure here and show how it works out in a particular example in the next section.

First assume that the neutral solution near the boundaries is given by

$$\phi_s = \{a_{1,2}(U(z)-c_s)^{\frac{1}{2}+\mu}+\dots\}, \quad 0<\mu<\tfrac{1}{2}. \tag{2.5}$$

Here $\mu = (\frac{1}{4}-R_{1,2})^{\frac{1}{2}}$, where $R = (J_s N^2/U'^2)$ with $R_1 = R(z_1)$ and $R_2 = R(z_2)$. $a_{1,2}$ is the amplitude near $z_{1,2}$. We assume that there exists an unstable solution ϕ contiguous to the neutral one. If $c-c_s$ is small enough, we find that near $z_{1,2}$

$$\phi = \{a_{1,2}[(U(z)-c)^{\frac{1}{2}+\mu}-(c_s-c)^{2\mu}(U(z)-c)^{\frac{1}{2}-\mu}]+\dots\}, \tag{2.6}$$

where we have used the fact that ϕ must be equal to zero at $z = z_{1,2}$. We see from (2.6) that, when $c\to c_s$, $\phi\to\phi_s$ for z near $z_{1,2}$. Let $\epsilon>0$ be chosen so small that in the intervals $(z_1, z_1+\epsilon)$ and $(z_2-\epsilon, z_2)$ ϕ_s is given by (2.5) and ϕ by (2.6). We write

$$I_1 = \int_{z_1}^{z_1+\epsilon}+\int_{z_1+\epsilon}^{z_2-\epsilon}+\int_{z_2-\epsilon}^{z_2}. \tag{2.7}$$

In this expression we first keep ϵ fixed and let $c-c_s$ tend to zero, then let ϵ tend to zero. When we keep ϵ fixed and let $c\to c_s$ we find that in the first and third integral on the right-hand side of (2.7) we have to use the expression (2.6) for ϕ. Into these integrals we introduce a new variable v given by $U(z)-c_s = (c-c_s)v$, and find that

$$\left.\begin{aligned}\int_{z_1}^{z_1+\epsilon} &\to J_s\left\{K_1(c-c_s)^{-1+2\mu}+\frac{2N_1^2}{2\mu-1}a_1^2(U_1')^{-2+2\mu}\epsilon^{-1+2\mu}\right\}\\ \int_{z_2-\epsilon}^{z_2} &\to J_s\left\{K_2(c-c_s)^{-1+2\mu}-\frac{2N_2^2}{2\mu-1}a_2^2(U_2')^{-2+2\mu}(-\epsilon)^{-1+2\mu}\right\}\end{aligned}\right\} \quad \text{as} \quad c\to c_s, \quad \epsilon\to 0. \tag{2.8}$$

Here $N_1 = N(z_1)$, $N_2 = N(z_2)$, $U_1' = U'(z_1)$ and $U_2' = U'(z_2)$. K_1 and K_2 are two constants given by two integrals which are easily found. Also

$$\int_{z_1+\epsilon}^{z_2-\epsilon}\to 2\int_{z_1+\epsilon}^{z_2-\epsilon}\frac{J_s N^2\phi_s^2}{(U-c_s)^3}dz \quad \text{as} \quad c\to c_s. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we obtain

$$I_1\to J_s(K_1+K_2)(c-c_s)^{-1+2\mu}+2\mathrm{P}\int_L\frac{J_s N^2\phi_s^2}{(U-c_s)^3}dz \quad \text{as} \quad c\to c_s, \tag{2.10}$$

where P in front of the integral sign means that the finite part of the integral is to be taken. Also

$$I_2\to\int_L\frac{U''\phi_s^2}{(U-c_s)^2}dz, \quad I_3\to\int_L\frac{N^2\phi_s^2}{(U-c_s)^2}dz, \quad I_4\to\int_L\phi_s^2 dz \quad \text{as} \quad c\to c_s. \tag{2.11}$$

We notice that the integrals in (2.11) become the same as the corresponding ones in Howard's formula, which was developed for a regular case with no critical layers at the boundaries. For the integral (2.10), however, there is a difference between these two cases.

For the integral I_1 in (2.10) we have the two possibilities:

(i) $K_1+K_2 = 0$, when the limiting value of I_1 is found by taking the finite part of the corresponding integral in Howard's formula;

(ii) $K_1+K_2 \neq 0$, when the dominant term is $J_s(K_1+K_2)(c-c_s)^{-1+2\mu}$.

By substituting (2.10) and (2.11) into (2.3) we can find $c-c_s$ as a function of $k^2-k_s^2$ when $k\to k_s$ with $J = J_s$ fixed or as a function of $J-J_s$ when $J\to J_s$ with $k = k_s$ fixed.

Now suppose that the neutral solution near the boundaries is given by

$$\phi_s = \{a_{1,2}(U(z)-c_s)^{\frac{1}{2}-\mu}+\dots\},$$

and assume that there exists an unstable solution ϕ contiguous to this neutral one. If ϕ is equal to zero at $z = z_{1,2}$, it must be given by

$$\phi = \{a_{1,2}[(U(z)-c)^{\frac{1}{2}-\mu}-(c_s-c)^{-2\mu}(U(z)-c)^{\frac{1}{2}+\mu}]+\dots\}.$$

But this means that ϕ does not tend to ϕ_s when $c \to c_s$ in contradiction of our assumption. We cannot have any unstable solution contiguous to this neutral one and application of Howard's formula has no meaning at all.

3. An example

We consider the case $U = \sin z$, $N^2 = 1$ and $-z_1 = z_2 = \pi$ (Huppert 1973), for which there exist the neutral solutions

$$\left.\begin{aligned} c_s = 0, \quad \phi_s = (\sin z)^{\frac{1}{2}\pm\mu}, \quad \text{where} \quad \mu = (\tfrac{1}{4}-J_s)^{\frac{1}{2}}, \\ J_s = (1-k_s^2)^{\frac{1}{2}}-1+k_s^2, \quad 0 \leqslant k_s \leqslant 1, \end{aligned}\right\} \tag{3.1}$$

where the plus sign is valid when $0 < k_s < \frac{1}{2}\sqrt{3}$ and the minus sign when $\frac{1}{2}\sqrt{3} < k_s < 1$, and (Thorpe 1969)

$$\left.\begin{aligned} c_s = 0, \quad \phi_s = (\cos \tfrac{1}{2}z)^{\frac{1}{2}\pm\mu}(\sin \tfrac{1}{2}z)^{\frac{1}{2}\mp\mu}, \quad \text{where} \quad \mu = (\tfrac{1}{4}-J_s)^{\frac{1}{2}}, \\ k_s = \tfrac{1}{2}\sqrt{3}, \quad 0 \leqslant J_s \leqslant \tfrac{1}{4}. \end{aligned}\right\} \tag{3.2}$$

The neutral solutions corresponding to the lower signs in (3.1) and (3.2) have no contiguous unstable solutions. This follows from the result at the end of § 2 and is in agreement with the results of Hazel (1972), who made a numerical investigation of this problem and found that these solutions are isolated neutral solutions.

First let the neutral solution be given by the one with the upper sign in (3.1). We put $k = k_s$ in (2.3) and evaluate the integrals I_1, I_2 and I_3, into which we substitute the expressions for U, N^2, ϕ_s and ϕ, where

$$\left.\begin{aligned} \phi_s &= (\pi-z)^{\frac{1}{2}+\mu} \\ \phi &= (\pi-z-c)^{\frac{1}{2}+\mu}-(-c)^{2\mu}(\pi-z-c)^{\frac{1}{2}-\mu} \end{aligned}\right\} \quad \text{near} \quad z=\pi,$$

$$\phi_s = \begin{cases} (\sin z)^{\frac{1}{2}+\mu} & \text{for} \quad 0<z<\pi, \\ e^{-i\pi(\frac{1}{2}+\mu)}[\sin(-z)]^{\frac{1}{2}+\mu} & \text{for} \quad -\pi<z<0, \end{cases}$$

$$\left.\begin{aligned} \phi_s &= e^{-i\pi(\frac{1}{2}+\mu)}(z+\pi)^{\frac{1}{2}+\mu} \\ \phi &= e^{-i\pi(\frac{1}{2}+\mu)}[(z+\pi+c)^{\frac{1}{2}+\mu}-c^{2\mu}(z+\pi+c)^{\frac{1}{2}-\mu}] \end{aligned}\right\} \quad \text{near} \quad z=-\pi.$$

When evaluating I_1 we find that $K_1 = -K_2 = e^{-2\pi i\mu}K$, where

$$K = \int_0^\infty v^{-\frac{3}{2}+\mu}\{(v+1)^{-\frac{3}{2}+\mu}-(v+1)^{-\frac{3}{2}-\mu}\}\{2v+1\}\,dv. \tag{3.3}$$

This is consequently an example where we should have obtained the correct result by taking the finite part of the corresponding diverging integral in Howard's formula.

Both this finite part and the values of I_2 and I_3 given by (2.11) are easily found, and we obtain

$$c(\mu+1)(1-2\mu)\cos\pi\mu B(\mu+\tfrac{1}{2},\tfrac{1}{2})+i(J-J_s)\sin\pi\mu B(\mu,\tfrac{1}{2}) = 0. \tag{3.4}$$

In this equation we have neglected terms of order $c^{2\mu+1}$ since they are small compared with c. However, these terms will account for the singular behaviour of the second derivative of c with respect to J which the numerical results of Banks *et al.* suggest. Equation (3.4) admits a pure imaginary solution $c = ic_i$ with $c_i > 0$ which is equivalent to the one obtained by Huppert (1973).

Next let the neutral solution be given by the one with the upper signs in (3.2). We put $J = J_s$ in (2.3) and evaluate the integrals I_1, I_2 and I_4, where now

$$\left.\begin{aligned}\phi_s &= (\tfrac{1}{2})^{\frac{1}{2}+\mu}(\pi-z)^{\frac{1}{2}+\mu}\\ \phi &= (\tfrac{1}{2})^{\frac{1}{2}+\mu}[(\pi-z-c)^{\frac{1}{2}+\mu}-(-c)^{2\mu}(\pi-z-c)^{\frac{1}{2}-\mu}]\end{aligned}\right\}\quad \text{near}\quad z=\pi,$$

$$\phi_s = \begin{cases}(\cos\tfrac{1}{2}z)^{\frac{1}{2}+\mu}(\sin\tfrac{1}{2}z)^{\frac{1}{2}-\mu} & \text{for}\quad 0<z<\pi,\\ e^{-i\pi(\frac{1}{2}-\mu)}(\cos\tfrac{1}{2}z)^{\frac{1}{2}+\mu}(\sin-\tfrac{1}{2}z)^{\frac{1}{2}-\mu} & \text{for}\quad -\pi<z<0,\end{cases}$$

$$\left.\begin{aligned}\phi_s &= e^{-i\pi(\frac{1}{2}-\mu)}(\tfrac{1}{2})^{\frac{1}{2}+\mu}(z+\pi)^{\frac{1}{2}+\mu}\\ \phi &= e^{-i\pi(\frac{1}{2}-\mu)}(\tfrac{1}{2})^{\frac{1}{2}+\mu}[(z+\pi+c)^{\frac{1}{2}+\mu}-c^{2\mu}(z+\pi+c)^{\frac{1}{2}-\mu}]\end{aligned}\right\}\quad \text{near}\quad z=-\pi.$$

In this case we find that $K_1 = (\frac{1}{2})^{1+2\mu}e^{2\pi i\mu}K$ and $K_2 = -(\frac{1}{2})^{1+2\mu}e^{-2\pi i\mu}K$, where K is given by (3.3), and that the finite part of the integral in (2.10) is zero. Substituting the expressions for I_1, I_2 and I_4 into (2.3) yields

$$iJ_s(\tfrac{1}{2})^{2\mu}\sin 2\pi\mu K c^{2\mu}+\pi e^{i\pi\mu}c+2\pi i\mu\, e^{i\pi\mu}(k^2-k_s^2) = 0. \tag{3.5}$$

Notice that in deriving (3.4) and (3.5) we have not made any assumption about the unstable solution being pure imaginary. However (3.5) does admit a pure imaginary solution $c = ic_i$ with $c_i > 0$. It can be shown that K has the value $2\mu/J_s$, so that the expression for c_i obtained from (3.5) equals the one given by Banks *et al.* and we have thus given a logical justification for their result.

4. Conclusion

In cases where there exist critical layers at the boundaries, Howard's formula is not valid because one of the integrals does not exist. However, provided that the finite part of this diverging integral is taken, the formula is shown to be applicable in some particular cases. This is true on part of the stability boundary in the case $U = \sin z$, $N^2 = 1$ with the boundaries at $z = \pm\pi$.

However, in general an equation of the form $ac^{2\mu}+bc+d(k^2-k_s^2)+e(J-J_s) = 0$ is valid, where a, b, d and e are constants. In the particular example mentioned above we obtain an expression for c equal to the one found by Banks *et al.* Our result is based on a method which is valid in this case, and we have consequently given a logical justification for the result of Banks *et al.*, which is based on Howard's formula.

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